Irreducible bilinear tensorial concomitants of an arbitrary complex bivector

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Irreducible bilinear tensorial concomitants of an arbitrary complex antisymmetric valence-2 tensor are derived in four-dimensional spacetime. In addition these bilinear concomitants are symmetric (or antisymmetric), self-dual (or anti-self-dual), and hermitian forms in the antisymmetric tensor. An important example of an antisymmetric valence-2 tensor, or bivector, is the electromagnetic field strength tensor which ordinarily is taken to be real-valued. In generalizing to complex-valued bivectors, the authors find the hermitian form versions of the well-known electromagnetic scalar invariants and stress-energy-momentum tensor, but also discover several novel tensors of total valence 2 and 4. These tensors have algebraic similarities to the Riemann, Weyl, and Ricci tensors.

I. INTRODUCTION

Bilinear, or second-order, concomitants of antisymmetric tensors of total valence 2, also known as bivectors, are of fundamental importance in electromagnetism and general relativity. In electromagnetism, see [1], the electromagnetic field strength is described by the normally real-valued tensor of total valence-2

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad \alpha,\beta = 0,1,2,3 \tag{1}$$

which is antisymmetric, that is, $F^{\alpha\beta} = -F^{\beta\alpha}$. All the bilinear tensorial concomitants of a real-valued $F^{\alpha\beta}$ of total valence two or less are well-known. They are: the two scalars

$$\mathcal{L}_+^{(r)} = \frac{1}{4} F^{\mu\nu} F_{\nu\mu} \tag{2}$$

$$\mathcal{L}_-^{(r)} = \frac{1}{8} F^{\mu\nu} F^{\alpha\beta} \epsilon_{\nu\mu\alpha\beta}, \tag{3}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor in four dimensions (with $\epsilon^{0123} = -1$), and the valence-2 tensor

$$T_{(r)}^{\alpha\beta} = F^{\alpha\mu} F_{\mu}{}^{\beta} - \frac{1}{4} \mathcal{L}_+^{(r)} g^{\alpha\beta} \tag{4}$$

where $g^{\alpha\beta}$ is the metric tensor. In electromagnetism, (2) is the Lagrangian density of the free electromagnetic field and (4) is the electromagnetic stress-energy-momentum tensor. In general relativity, the stress-energy-momentum tensor plays a crucial role in the Einstein-Maxwell "electrovac" equations, see [2].

Usually bivectors are taken to be real-valued, and in this case the bilinear concomitants $\mathcal{L}_+^{(r)}$, $\mathcal{L}_-^{(r)}$, $T_{(r)}^{\alpha\beta}$ are the only possible concomitants tensors of total valence two or less. However, if we consider the bivector to be complex-valued, it is possible to construct an entirely new set of bilinear tensorial concomitants which have previously not been published. There are several reasons why one should consider arbitrary complex bivectors. For quantum radiation, the field strength operators of the electromagnetic field are necessarily complex and correspond directly to electromagnetic field observables, see [3]; and also for classical radiation it is convenient to treat wave-like phenomena using complex field variables, see [4]. The imaginary part of the bivector can, for instance, be constructed by taking the Hilbert transform of the real-valued fields, and the result is known as the analytical signal representation. We will, however, not use any properties particular to analytic signals, only that the bivector is arbitrarily complex. Curiously, the term "bivector" seems to have been originally used by Gibbs [5] to denote complex-valued 3-vectors.

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Thus the purpose of this paper is to explore the bilinear tensorial concomitants of an arbitrary complex bivector. First, we construct a set of valence-4 tensor concomitants bilinear in $F^{\alpha\beta}$ from which all others can be created through appropriate linear combinations and contractions; these are contracted to find all tensors of total valence 2 and 0; then we construct irreducible tensors for all valences; and then ultimately we assemble a complete set of irreducible tensors. All these tensor are such that they are hermitian forms in the bivector, invariant up to a sign under a duality transformation of the bivector, either symmetric or antisymmetric, and irreducible. Thus by construction, these tensors have properties that match those possessed by (2), (3), and (4). Although we are particularly interested in the concomitants of the Maxwell bivector, we will keep the treatment general so it can be applied to any covariant complex bivector. Finally examining the constructed irreducible bilinear tensor concomitants we find that several of them are completely novel.

This work is related to Olofsson [6] which attempts to decompose valence-4 bilinear concomitants of complex bivectors. An alternative approach to constructing bilinear concomitants of complex bivectors using tensor calculus as used here, is to use spinor calculus. A paper using a spinor approach is in preparation [7]. The tensors constructed here are also indirectly related to the Riemann, Weyl, and Ricci tensors, in that they share some algebraic properties.

II. CRITERIA FOR THE CONCOMITANTS AND BASIC ASSUMPTIONS

In what follows, we will consider an arbitrary (in general not self-dual) contravariant complex bivector $F^{\alpha\beta}$. That is, we take $F^{\alpha\beta}$ to fulfill

$$F^{\alpha\beta} = -F^{\beta\alpha}, \quad \text{where } F^{\alpha\beta} \in \mathbb{C} \tag{5}$$

for all $\alpha, \beta = 0, 1, 2, 3$ (Greek indices run over 0,1,2,3 where 0 is the time-like dimension and the rest are space-like).

A bivector in spacetime can be constructed from two 3-vectors: one being the space-time part of the corresponding valence-2 tensor, the other the time-space part. That is, for any two complex 3-vectors

$$\mathbf{E} = (E_1, E_2, E_3)^{\mathsf{T}} \tag{6}$$

$$\mathbf{B} = (B_1, B_2, B_3)^{\mathsf{T}}, \tag{7}$$

where T denotes the transpose operator, the tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\mathbf{E}^{\mathsf{T}} \\ \mathbf{E} & \mathbf{B}\times \end{pmatrix} \tag{8}$$

is an arbitrary complex bivector, where $\mathbf{B}\times = \epsilon_{ijk} B_j$ and ϵ_{ijk} is the Levi-Civita tensor in three dimensions (lowercase italic letters $i, j, k = 1, 2, 3$ represent Cartesian components in 3-space). A complex bivector is easily seen to have 12 real-valued degrees of freedom in general.

Since $F^{\alpha\beta}$ in general has an imaginary part and we wish to construct hermitian form concomitants, we need to consider the complex conjugate of the bivector. We denote the complex conjugation of a tensor $\bar{F}^{\alpha\beta}$, and define it as

$$\bar{F}^{\alpha\beta} := \Re\left\{F^{\alpha\beta}\right\} - \mathbf{i}\Im\left\{F^{\alpha\beta}\right\}, \tag{9}$$

where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote the real part and imaginary part respectively of its argument. Naturally, the complex conjugate of the bivector is also antisymmetric so $\bar{F}^{\alpha\beta} = -\bar{F}^{\beta\alpha}$.

The (contravariant) dual of $F^{\alpha\beta}$ is defined as

$$^*F^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu} F^{\mu\nu}. \tag{10}$$

If the dual is viewed as a transform mapping bivectors to bivectors, we can define the duality transform to be

$$^*: \quad F^{\alpha\beta} \mapsto {}^*F^{\alpha\beta},\ {}^*F^{\alpha\beta} \mapsto -F^{\alpha\beta},\ \bar{F}^{\alpha\beta} \mapsto {}^*\bar{F}^{\alpha\beta},\ {}^*\bar{F}^{\alpha\beta} \mapsto -\bar{F}^{\alpha\beta}, \tag{11}$$

which is equivalent to exchanging the bivectors 3-vector components according to $\mathbf{E} \mapsto -\mathbf{B}$ and $\mathbf{B} \mapsto \mathbf{E}$.

As with real-valued bivectors, it is possible to construct from an arbitrary complex bivector a bivector which is invariant up to a factor $\pm\mathbf{i}$ under the duality transform, namely

$$^-\mathcal{F}^{\alpha\beta} = \left(F^{\alpha\beta} + \mathbf{i}\,{}^*F^{\alpha\beta}\right)/2 \tag{12}$$

$$^+\mathcal{F}^{\alpha\beta} = \left(F^{\alpha\beta} - \mathbf{i}\,{}^*F^{\alpha\beta}\right)/2 \tag{13}$$

$$\left(\overline{^-\mathcal{F}}\right)^{\alpha\beta} = \left(\bar{F}^{\alpha\beta} - \mathbf{i}\,{}^*\bar{F}^{\alpha\beta}\right)/2 \tag{14}$$

$$\left(\overline{^+\mathcal{F}}\right)^{\alpha\beta} = \left(\bar{F}^{\alpha\beta} + \mathbf{i}\,{}^*\bar{F}^{\alpha\beta}\right)/2, \tag{15}$$

see [2]. The $+$ and $-$ left superscripts denote the sign of the eigenvalue, $+\mathbf{i}$ or $-\mathbf{i}$ respectively, under the duality transformation. Bivectors with eigenvalue $+\mathbf{i}$ under a duality transformation are called self-dual and those with $-\mathbf{i}$ are called anti-self-dual. Compared to the case of real bivectors there is one new self-dual bivector, Eq. (14), and one new anti-self-dual bivector, Eq. (15). Note that contrary to the case of real $F^{\alpha\beta}$, the complex conjugate of the self-dual bivector associated with a complex $F^{\alpha\beta}$ does not, in general, give the anti-self-dual nor vice-versa, that is $\left(\overline{{}^{+}\mathcal{F}}\right)^{\alpha\beta} \neq {}^{-}\mathcal{F}^{\alpha\beta}$ and $\left(\overline{{}^{-}\mathcal{F}}\right)^{\alpha\beta} \neq {}^{+}\mathcal{F}^{\alpha\beta}$.

To be clear, a tensorial concomitant of $F^{\alpha\beta}$ is an algebraic combination of $F^{\alpha\beta}$, $g^{\gamma\delta}$, or $\epsilon^{\mu\nu\alpha\beta}$, possibly with contracted indices. A bilinear tensorial concomitant is a tensorial concomitant that is second-order in $F^{\alpha\beta}$. Out of the possible bilinear tensorial concomitants involving complex bivectors we will only be interested in those that are hermitian forms in $F^{\alpha\beta}$. By hermitian form we mean a bilinear form that is invariant under the multiplication of $F^{\alpha\beta}$ by an arbitrary phase factor, that is, a hermitian form in $F^{\alpha\beta}$ is unaltered under the transformations

$$F^{\alpha\beta} \mapsto \exp(+\mathbf{i}\phi)F^{\alpha\beta}, \quad \bar{F}^{\alpha\beta} \mapsto \exp(-\mathbf{i}\phi)\bar{F}^{\alpha\beta}. \tag{16}$$

For example, tensors such as $\bar{F}^{\alpha\beta}F^{\gamma\delta}$ or $\bar{F}^{\alpha\beta}F^{\gamma\delta}\epsilon_{\gamma\delta\mu\nu}$ are hermitian form tensorial concomitants in $F^{\alpha\beta}$.

In addition, the concomitants we seek should be irreducible tensors. By irreducible tensor we mean that the tensor cannot be decomposed into tensors of lower total valence. For valence-2 tensors in four dimensional space this means that the tensor should be trace-free, that is, it should vanish when fully contracted with the metric tensor

$$T_{\mu}{}^{\mu} = 0. \tag{17}$$

For valence-4 tensors in four dimensions, irreducibility means that the tensor should be trace-free over all pairs of indices and in addition vanish when fully contracted with the Levi-Civita tensor. Thus, for an arbitrary valence-4 tensor $\mathcal{T}^{\alpha\beta\gamma\delta}$ to be irreducible it must satisfy all of the following conditions

$$T_{\mu}{}^{\mu\alpha\beta} = T_{\mu}{}^{\alpha\mu\beta} = T^{\alpha}{}_{\mu}{}^{\mu\beta} = T^{\alpha}{}_{\mu}{}^{\beta\mu} = T^{\alpha\beta}{}_{\mu}{}^{\mu} = T_{\mu}{}^{\alpha\beta\mu} = 0 \tag{18}$$

$$\epsilon_{\mu\nu\alpha\beta}T^{\mu\nu\alpha\beta} = 0 \tag{19}$$

$$\epsilon_{\alpha\mu\nu\gamma}T^{\beta\mu\nu\gamma} = \epsilon_{\alpha\mu\nu\gamma}T^{\mu\beta\nu\gamma} = \epsilon_{\alpha\mu\nu\gamma}T^{\mu\nu\beta\gamma} = \epsilon_{\alpha\mu\nu\gamma}T^{\mu\nu\gamma\beta} = 0. \tag{20}$$

To be precise, the tensors we seek are actually real-irreducible tensors or Cartesian tensors as they are sometimes known, see [8]. By this we mean that the tensors fulfill the irreducibility conditions above and that their basis vectors are real-valued. In what follows, we will sometimes simply use the term "irreducible tensor" since we will not be dealing with complex base vectors, or spherical tensors as they are also known.

In addition to the above mentioned criteria, the sought after tensorial concomitants should also be either symmetric or antisymmetric.

Let us recapitulate the objective of this paper: we seek bilinear concomitants of an arbitrary complex bivector $F^{\alpha\beta}$ that are

- hermitian forms in the components of $F^{\alpha\beta}$: invariant under (16),
- (anti)-self-dual: invariant up to sign under (11),
- symmetric or anti-symmetric,
- irreducible: fulfilling either (17) or (18), (19), (20).

These conditions have been chosen so that the constructed bilinear concomitants of complex $F^{\alpha\beta}$ are consistent with the properties possessed by the well-known bilinear concomitants of real $F^{\alpha\beta}$.

Our plan on how to construct the concomitants is as follows: we take all possible hermitian form outer products of the four self-dual tensors, ${}^{-}\mathcal{F}_{\gamma\delta}$, ${}^{+}\mathcal{F}_{\gamma\delta}$, $\left(\overline{{}^{-}\mathcal{F}}\right)_{\alpha\beta}$ and $\left(\overline{{}^{+}\mathcal{F}}\right)_{\alpha\beta}$, which constitute all possible valence-4 tensor combinations; from these tensors we take all possible contractions over two indices to obtain all valence-2 concomitants; and then we contracted over the finally pair of indices to obtain all the scalar concomitants. From these tensors of total valence 0, 2, and 4 we construct irreducible tensors which are finally assembled into an irreducible tensorial set.

As for the metric, we assume throughout that the metric tensor is real-valued and symmetric and that in a local coordinate system the metric can be set to the Lorentzian metric for which $g^{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$.

III. HERMITIAN FORM SELF DUAL CONCOMITANTS OF VALENCE FOUR

We seek to construct bilinear concomitants of the complex bivector $F_{\alpha\beta}$ which are hermitian forms and are invariant up to a sign under the duality transformation of the bivector. To this end we take all bilinear combinations of the complex conjugate self-dual bivectors $\overline{({}^{-}\mathcal{F})}_{\alpha\beta}$ and $\overline{({}^{+}\mathcal{F})}_{\alpha\beta}$ with the self-dual bivectors ${}^{-}\mathcal{F}_{\gamma\delta}$ and ${}^{+}\mathcal{F}_{\gamma\delta}$. This gives the four combinations

$$4\overline{({}^{-}\mathcal{F})}_{\alpha\beta}{}^{-}\mathcal{F}_{\gamma\delta} = \bar{F}_{\alpha\beta}F_{\gamma\delta} + {}^{*}\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} + \mathbf{i}\left(\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} - {}^{*}\bar{F}_{\alpha\beta}F_{\gamma\delta}\right) \tag{21}$$

$$4\overline{({}^{+}\mathcal{F})}_{\alpha\beta}{}^{+}\mathcal{F}_{\gamma\delta} = \bar{F}_{\alpha\beta}F_{\gamma\delta} + {}^{*}\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} - \mathbf{i}\left(\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} - {}^{*}\bar{F}_{\alpha\beta}F_{\gamma\delta}\right) \tag{22}$$

$$4\overline{({}^{-}\mathcal{F})}_{\alpha\beta}{}^{+}\mathcal{F}_{\gamma\delta} = \bar{F}_{\alpha\beta}F_{\gamma\delta} - {}^{*}\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} - \mathbf{i}\left(\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} + {}^{*}\bar{F}_{\alpha\beta}F_{\gamma\delta}\right) \tag{23}$$

$$4\overline{({}^{+}\mathcal{F})}_{\alpha\beta}{}^{-}\mathcal{F}_{\gamma\delta} = \bar{F}_{\alpha\beta}F_{\gamma\delta} - {}^{*}\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} + \mathbf{i}\left(\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} + {}^{*}\bar{F}_{\alpha\beta}F_{\alpha\delta}\right). \tag{24}$$

These are all tensors of valence-4, self-dual, and manifestly hermitian forms.

However, in order to simplify the proceeding and ultimately obtain the complex analogs of the well known bilinear concomitants of real-valued bivectors, we instead take quantities proportional to the sum and difference of (21) and (22), and proportional to the sum and difference of (23) and (24). Specifically, we introduce the following four tensors

$$T'_{\alpha\beta\gamma\delta} := \overline{({}^{-}\mathcal{F})}_{\alpha\beta}{}^{-}\mathcal{F}_{\gamma\delta} + \overline{({}^{+}\mathcal{F})}_{\alpha\beta}{}^{+}\mathcal{F}_{\gamma\delta} = \left(\bar{F}_{\alpha\beta}F_{\gamma\delta} + {}^{*}\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta}\right)/2 \tag{25}$$

$$Q'_{\alpha\beta\gamma\delta} := \overline{({}^{-}\mathcal{F})}_{\alpha\beta}{}^{-}\mathcal{F}_{\gamma\delta} - \overline{({}^{+}\mathcal{F})}_{\alpha\beta}{}^{+}\mathcal{F}_{\gamma\delta} = \mathbf{i}\left(\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} - {}^{*}\bar{F}_{\alpha\beta}F_{\gamma\delta}\right)/2 \tag{26}$$

$$D'_{\alpha\beta\gamma\delta} := \overline{({}^{-}\mathcal{F})}_{\alpha\beta}{}^{+}\mathcal{F}_{\gamma\delta} + \overline{({}^{+}\mathcal{F})}_{\alpha\beta}{}^{-}\mathcal{F}_{\gamma\delta} = \left(\bar{F}_{\alpha\beta}F_{\gamma\delta} - {}^{*}\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta}\right)/2 \tag{27}$$

$$X'_{\alpha\beta\gamma\delta} := \mathbf{i}\left(\overline{({}^{-}\mathcal{F})}_{\alpha\beta}{}^{+}\mathcal{F}_{\gamma\delta} - \overline{({}^{+}\mathcal{F})}_{\alpha\beta}{}^{-}\mathcal{F}_{\gamma\delta}\right) = \left(\bar{F}_{\alpha\beta}{}^{*}F_{\gamma\delta} + {}^{*}\bar{F}_{\alpha\beta}F_{\gamma\delta}\right)/2. \tag{28}$$

These valence-4 tensors can be expressed in terms of the two 3-vector components of the bivector, $\mathbf{E}$ and $\mathbf{B}$, by using so-called bivector indexing, see [9]. A bivector index is denoted with uppercase roman letters A and B which run through values 1 to 6. F^{A} is used to represent the tensor component $F^{\alpha\beta}$ where the index mapping $A \leftrightarrow [\alpha\beta]$ is taken to be $1 \leftrightarrow [10]$, $2 \leftrightarrow [20]$, $3 \leftrightarrow [30]$, $4 \leftrightarrow [32]$, $5 \leftrightarrow [13]$, $6 \leftrightarrow [21]$. This maps the valence-2 antisymmetric tensor, as given in (8), according to

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\mathbf{E}^{\mathsf{T}} \\ \mathbf{E} & \mathbf{B}\times \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = F^{A} \tag{29}$$

where the vector with six components on the left-hand side, F^{A}, is known as a sixtor, see [10]. The valence-4 tensors can therefore be written in terms of the two 3-vectors (6) and (7) using the matrices

$$T'^{\alpha\beta\gamma\delta} \leftrightarrow T'^{AB} = \frac{1}{2}\begin{pmatrix} \bar{\mathbf{E}}\otimes\mathbf{E} + \bar{\mathbf{B}}\otimes\mathbf{B} & \bar{\mathbf{E}}\otimes\mathbf{B} - \bar{\mathbf{B}}\otimes\mathbf{E} \\ -\left(\bar{\mathbf{E}}\otimes\mathbf{B} - \bar{\mathbf{B}}\otimes\mathbf{E}\right) & \bar{\mathbf{E}}\otimes\mathbf{E} + \bar{\mathbf{B}}\otimes\mathbf{B} \end{pmatrix} \tag{30}$$

$$Q'^{\alpha\beta\gamma\delta} \leftrightarrow Q'^{AB} = \frac{\mathbf{i}}{2}\begin{pmatrix} -\left(\bar{\mathbf{E}}\otimes\mathbf{B} - \bar{\mathbf{B}}\otimes\mathbf{E}\right) & \bar{\mathbf{E}}\otimes\mathbf{E} + \bar{\mathbf{B}}\otimes\mathbf{B} \\ -\left(\bar{\mathbf{E}}\otimes\mathbf{E} + \bar{\mathbf{B}}\otimes\mathbf{B}\right) & -\left(\bar{\mathbf{E}}\otimes\mathbf{B} - \bar{\mathbf{B}}\otimes\mathbf{E}\right) \end{pmatrix} \tag{31}$$

$$D'^{\alpha\beta\gamma\delta} \leftrightarrow D'^{AB} = \frac{1}{2}\begin{pmatrix} \bar{\mathbf{E}}\otimes\mathbf{E} - \bar{\mathbf{B}}\otimes\mathbf{B} & \bar{\mathbf{E}}\otimes\mathbf{B} + \bar{\mathbf{B}}\otimes\mathbf{E} \\ \bar{\mathbf{E}}\otimes\mathbf{B} + \bar{\mathbf{B}}\otimes\mathbf{E} & -\left(\bar{\mathbf{E}}\otimes\mathbf{E} - \bar{\mathbf{B}}\otimes\mathbf{B}\right) \end{pmatrix} \tag{32}$$

$$X'^{\alpha\beta\gamma\delta} \leftrightarrow X'^{AB} = \frac{1}{2}\begin{pmatrix} -\left(\bar{\mathbf{E}}\otimes\mathbf{B} + \bar{\mathbf{B}}\otimes\mathbf{E}\right) & \bar{\mathbf{E}}\otimes\mathbf{E} - \bar{\mathbf{B}}\otimes\mathbf{B} \\ \bar{\mathbf{E}}\otimes\mathbf{E} - \mathbf{B}\otimes\mathbf{B} & \bar{\mathbf{E}}\otimes\mathbf{B} + \bar{\mathbf{B}}\otimes\mathbf{E} \end{pmatrix} \tag{33}$$

where $\bar{\mathbf{E}}\otimes\mathbf{E} = \bar{E}_i E_j$ is the outer or tensor product between $\bar{\mathbf{E}}$ and $\mathbf{E}$. As is clear from these sixtor matrices, the four valence-4 tensors $T'^{\alpha\beta\gamma\delta}$, $Q'^{\alpha\beta\gamma\delta}$, $D'^{\alpha\beta\gamma\delta}$, and $X'^{\alpha\beta\gamma\delta}$ together represent all hermitian forms of valence-4 in the components of the bivector. Each of these tensors has 18 independent components. In fact, the tensor pair $T'^{\alpha\beta\gamma\delta}$ and $Q'^{\alpha\beta\gamma\delta}$ or the pair $D'^{\alpha\beta\gamma\delta}$ and $X'^{\alpha\beta\gamma\delta}$ on their own constitute an independent and complete set of 36 hermitian forms

in the components of the complex bivector counting real and imaginary parts separately. This corresponds to the **36** possible hermitian forms in the components of the complex bivector counting real and imaginary parts separately.

The tensors defined above, $T'^{\alpha\beta\gamma\delta}$, $Q'^{\alpha\beta\gamma\delta}$, $D'^{\alpha\beta\gamma\delta}$, $X'^{\alpha\beta\gamma\delta}$, all have the following properties

$$T'^{\alpha\beta\gamma\delta} = -T'^{\beta\alpha\gamma\delta} = -T'^{\alpha\beta\delta\gamma} = T'^{\beta\alpha\delta\gamma} \tag{34}$$

$$T'^{\alpha\beta\gamma\delta} = \bar{T}'^{\gamma\delta\alpha\beta} \tag{35}$$

where we have used $T'^{\alpha\beta\gamma\delta}$ as an example for any of the four tensors.

IV. VALENCE TWO CONCOMITANTS OF SELF DUAL BIVECTORS

Having derived the general valence-4 tensors, we can now construct valence-2 tensors by simply contracting the valence-4 tensors over pairs of indices. Six such contractions are possible, but from the symmetry relations (34) it is clear that the contraction between the indices (1,4), (2,3), (1,3), and (2,4) are all the same up to a sign. Furthermore, the contraction between indices (1,2) and (3,4) are zero. Therefore the all non-vanishing contractions are, up to a sign, given by

$$T^{\alpha\beta} := T'^{\alpha\mu\nu\beta} g_{\mu\nu} = \left(\bar{F}^{\alpha}{}_{\mu} F^{\mu\beta} + {}^*\bar{F}^{\alpha}{}_{\mu} {}^*F^{\mu\beta}\right)/2 \tag{36}$$

$$Q^{\alpha\beta} := Q'^{\alpha\mu\nu\beta} g_{\mu\nu} = \mathbf{i}\left(\bar{F}^{\alpha}{}_{\mu} {}^*F^{\mu\beta} - {}^*\bar{F}^{\alpha}{}_{\mu} F^{\mu\beta}\right)/2 \tag{37}$$

$$D'^{\alpha\beta} := D'^{\alpha\mu\nu\beta} g_{\mu\nu} = \left(\bar{F}^{\alpha}{}_{\mu} F^{\mu\beta} - {}^*\bar{F}^{\alpha}{}_{\mu} {}^*F^{\mu\beta}\right)/2 \tag{38}$$

$$X'^{\alpha\beta} := X'^{\alpha\mu\nu\beta} g_{\mu\nu} = \left(\bar{F}^{\alpha}{}_{\mu} {}^*F^{\mu\beta} + {}^*\bar{F}^{\alpha}{}_{\mu} F^{\mu\beta}\right)/2. \tag{39}$$

All these tensors have total valence two. Furthermore, $T^{\alpha\beta}$ and $Q^{\alpha\beta}$ are symmetric,

$$T^{\alpha\beta} = T^{\beta\alpha} \tag{40}$$

$$Q^{\alpha\beta} = Q^{\beta\alpha}, \tag{41}$$

real-valued

$$\Im\left\{T^{\alpha\beta}\right\} = 0 \tag{42}$$

$$\Im\left\{Q^{\alpha\beta}\right\} = 0 \tag{43}$$

and trace-free as will be shown in the next section. $D'^{\alpha\beta}$ and $X'^{\alpha\beta}$, on the other hand, are neither symmetric nor antisymmetric, have both real and imaginary parts, and have are not trace-free. Hence $T^{\alpha\beta}$ and $Q^{\alpha\beta}$ irreducible while $D'^{\alpha\beta}$ and $X'^{\alpha\beta}$ are not. This is the reason for the prime accents ($'$) on $D'^{\alpha\beta}$ and $X'^{\alpha\beta}$: they mark the fact that these tensors are not irreducible.

In a local coordinate system, we can set the metric tensor to be Lorentzian. In this case the two valence-2 tensors $T^{\alpha\beta}$ and $Q^{\alpha\beta}$ can be written in terms of the two complex 3-vector components of the bivector as follows

$$T^{00} = \left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)/2 \tag{44}$$

$$T^{i0} = \Re\left\{\bar{\mathbf{E}} \times \mathbf{B}\right\} \tag{45}$$

$$T^{ij} = -\Re\left\{\bar{\mathbf{E}} \otimes \mathbf{E} + \bar{\mathbf{B}} \otimes \mathbf{B}\right\} + T^{00}\mathbf{1}_3 \tag{46}$$

$$T^{ji} = T^{ij} \tag{47}$$

$$Q^{00} = \Im\left\{\bar{\mathbf{E}} \cdot \mathbf{B}\right\} \tag{48}$$

$$Q^{i0} = \frac{\mathbf{i}}{2}\left(\bar{\mathbf{E}} \times \mathbf{E} + \bar{\mathbf{B}} \times \mathbf{B}\right) \tag{49}$$

$$Q^{ij} = -\Im\left\{\bar{\mathbf{E}} \otimes \mathbf{B} - \bar{\mathbf{B}} \otimes \mathbf{E}\right\} + Q^{00}\mathbf{1}_3 \tag{50}$$

$$Q^{ji} = Q^{ij} \tag{51}$$

where $\mathbf{1}_3$ is the unity valence-2 tensor in 3-space.

Physically, if we take $F^{\alpha\beta}$ to be the electromagnetic field strength tensor, then the symmetric tensor $T^{\alpha\beta}/4\pi$ can be interpreted as the hermitian form generalization of the stress-energy-momentum tensor in Gaussian units, see [1]. Parts of this hermitian generalization can be found in the time-harmonic analysis of electromagnetic energy such as in the complex Poynting theorem, see [1]. The other valence-2 tensors lack, as yet, a specific physical interpretation. Having said that, the real-valued 3-vector in the time-space components of $Q^{\alpha\beta}$, namely $Q^{i0} = \frac{\mathbf{i}}{2}\left(\bar{\mathbf{E}}\times\mathbf{E}+\bar{\mathbf{B}}\times\mathbf{B}\right)$, is proportional to the vector mentioned in the beginning of Lipkin [11] and related to the vector defined in equation (17) in Carozzi [12]. Based on this relationship one can argue that $Q^{\alpha\beta}$ is the spin-weighted stress-energy-momentum tensor of the electromagnetic field.

V. SCALAR CONCOMITANTS

The valence-0, or scalar, concomitants can now be found by contracting the valence-2 tensors. Of the two possible ways of contracting both are the same since we are assuming the metric tensor is symmetric. Thus all possible traces are given by

$$
\begin{aligned}
0 =&T^{\mu\nu}g_{\mu\nu} = \left(\bar{F}_{\mu\nu}F^{\nu\mu} + {}^*\bar{F}_{\mu\nu}{}^*F^{\nu\mu}\right)/2 && (52)\\
0 =&Q^{\mu\nu}g_{\mu\nu} = \mathbf{i}\left(\bar{F}_{\mu\nu}{}^*F^{\nu\mu} - {}^*\bar{F}_{\mu\nu}F^{\nu\mu}\right)/2 && (53)\\
\mathcal{L}_+ =&D'^{\mu\nu}g_{\mu\nu}/4 = \left(\bar{F}_{\mu\nu}F^{\nu\mu} - {}^*\bar{F}_{\mu\nu}{}^*F^{\nu\mu}\right)/2 && (54)\\
\mathcal{L}_- =&X'^{\mu\nu}g_{\mu\nu}/4 = \left(\bar{F}_{\mu\nu}{}^*F^{\nu\mu} + {}^*\bar{F}_{\mu\nu}F^{\nu\mu}\right)/2. && (55)
\end{aligned}
$$

Thus, the tensors $T^{\alpha\beta}$ and $Q^{\alpha\beta}$ are trace-free while the trace of $D'^{\alpha\beta}$ and $X'^{\alpha\beta}$ leads to two scalars which are the hermitian form generalization of the well-known scalar invariants, (2) and (3).

In a local coordinate system the two scalars $\mathcal{L}_+$ and $\mathcal{L}_-$ can be written in terms of the 3-vector components of the bivector as follows

$$
\begin{aligned}
\mathcal{L}_+ =&\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right)/2 && (56)\\
\mathcal{L}_- =&-\Re\left\{\bar{\mathbf{E}}\cdot\mathbf{B}\right\}. && (57)
\end{aligned}
$$

When the bivector represents a complex-valued electromagnetic field, $\mathcal{L}_+/4\pi$ is the hermitian form generalization of the flat vacuum electromagnetic field Lagrangian density in Gaussian units and $\mathcal{L}_-$ is the hermitian form version of the pseudo-scalar invariant of the electromagnetic field in flat spacetime. Their subscript sign indicates their eigenvalue under the duality transformation which is either $+1$ or -1.

VI. IRREDUCIBLE CONCOMITANTS OF VALENCE TWO

As mentioned in the previous section, the tensors $T^{\alpha\beta}$ and $Q^{\alpha\beta}$ are irreducible while the tensors $D'^{\alpha\beta}$ and $X'^{\alpha\beta}$ are not. A new pair of tensor can be constructed in which the trace of $D'^{\alpha\beta}$ and $X'^{\alpha\beta}$ is removed

$$
\begin{aligned}
D^{\alpha\beta} :=& -\mathbf{i}\left(D'^{\alpha\beta} - \mathcal{L}_+ g^{\alpha\beta}\right) && (58)\\
X^{\alpha\beta} :=& -\mathbf{i}\left(X'^{\alpha\beta} - \mathcal{L}_- g^{\alpha\beta}\right). && (59)
\end{aligned}
$$

Obviously they are trace-free by construction,

$$
\begin{aligned}
D^{\mu}{}_{\mu} =&0 && (60)\\
X^{\mu}{}_{\mu} =&0, && (61)
\end{aligned}
$$

so $D^{\alpha\beta}$ and $X^{\alpha\beta}$ are irreducible tensors of total valence two. The tensors $D^{\alpha\beta}$ and $X^{\alpha\beta}$ are also equal to the imaginary parts of $D'^{\alpha\beta}$ and $X'^{\alpha\beta}$ respectively, that is

$$
\begin{aligned}
D^{\alpha\beta} =&\Im\left\{D'^{\alpha\beta}\right\} && (62)\\
X^{\alpha\beta} =&\Im\left\{X'^{\alpha\beta}\right\}. && (63)
\end{aligned}
$$

and so $D^{\alpha\beta}$ and $X^{\alpha\beta}$ are purely real,

$$\Im\left\{D^{\alpha\beta}\right\} =0 \tag{64}$$
$$\Im\left\{X^{\alpha\beta}\right\} =0. \tag{65}$$

However, in contrast to $T^{\alpha\beta}$ and $Q^{\alpha\beta}$ which are symmetric, $D^{\alpha\beta}$ and $X^{\alpha\beta}$ are antisymmetric,

$$D^{\alpha\beta} =-\, D^{\beta\alpha} \tag{66}$$
$$X^{\alpha\beta} =-\, X^{\beta\alpha}. \tag{67}$$

Furthermore, $D^{\alpha\beta}$ and $X^{\alpha\beta}$ are mutually dual, that is

$$^{*}D^{\alpha\beta} = +X^{\alpha\beta}, \quad ^{*}X^{\alpha\beta} = -D^{\alpha\beta} \tag{68}$$

while $T^{\alpha\beta}$ and $Q^{\alpha\beta}$ are not mutually dual. In fact, their duals are zero,

$$^{*}T^{\alpha\beta} =0 \tag{69}$$
$$^{*}Q^{\alpha\beta} =0. \tag{70}$$

In a local, Lorentzian metric, coordinate system, the components of $D^{\alpha\beta}$ and $X^{\alpha\beta}$ are

$$D^{00} =0 \tag{71}$$
$$D^{i0} =-\,\Im\left\{\bar{\mathbf{E}}\times\mathbf{B}\right\} \tag{72}$$
$$D^{ij} =-\,\Im\left\{\bar{\mathbf{E}}\otimes\mathbf{E}-\bar{\mathbf{B}}\otimes\mathbf{B}\right\} = -\frac{\mathbf{i}}{2}\left(\bar{\mathbf{E}}\times\mathbf{E}-\bar{\mathbf{B}}\times\mathbf{B}\right)\times \tag{73}$$
$$D^{ji} =-\, D^{ij} \tag{74}$$

$$X^{00} =0 \tag{75}$$
$$X^{i0} =\frac{\mathbf{i}}{2}\left(\bar{\mathbf{E}}\times\mathbf{E}-\bar{\mathbf{B}}\times\mathbf{B}\right) \tag{76}$$
$$X^{ij} =\Im\left\{\bar{\mathbf{E}}\otimes\mathbf{B}+\bar{\mathbf{B}}\otimes\mathbf{E}\right\} = \frac{\mathbf{i}}{2}\left(\bar{\mathbf{E}}\times\mathbf{B}-\mathbf{E}\times\bar{\mathbf{B}}\right)\times \tag{77}$$
$$X^{ji} =-\, X^{ij} \tag{78}$$

where we have used the 3-vector components of $F^{\alpha\beta}$. Being mutually dual means that $D^{\alpha\beta}$ and $X^{\alpha\beta}$ together contain the same information as the two real 3-vectors $\Im\left\{\bar{\mathbf{E}}\times\mathbf{B}\right\}$ and $\mathbf{i}\left(\bar{\mathbf{E}}\times\mathbf{E}-\bar{\mathbf{B}}\times\mathbf{B}\right)/2$. The vector $\Im\left\{\bar{\mathbf{E}}\times\mathbf{B}\right\}$ is recognized to be proportional to the imaginary part of the complex Poynting vector, see [1].

VII. IRREDUCIBLE CONCOMITANTS OF VALENCE FOUR

The two tensors, $T'^{\alpha\beta\gamma\delta}$ and $Q'^{\alpha\beta\gamma\delta}$ are completely reducible as follows

$$T'^{\alpha\beta\gamma\delta} =2T^{[\alpha[\delta}g^{\gamma]\beta]} - \frac{\mathbf{i}}{4}\left(T^{\alpha\mu}\epsilon_{\mu}{}^{\beta\gamma\delta} - T^{\beta\mu}\epsilon_{\mu}{}^{\beta\gamma\delta\alpha} - T^{\gamma\mu}\epsilon_{\mu}{}^{\delta\alpha\beta} + T^{\delta\mu}\epsilon_{\mu}{}^{\alpha\beta\gamma}\right) \tag{79}$$
$$Q'^{\alpha\beta\gamma\delta} =2Q^{[\alpha[\delta}g^{\gamma]\beta]} - \frac{\mathbf{i}}{4}\left(Q^{\alpha\mu}\epsilon_{\mu}{}^{\beta\gamma\delta} - Q^{\beta\mu}\epsilon_{\mu}{}^{\beta\gamma\delta\alpha} - Q^{\gamma\mu}\epsilon_{\mu}{}^{\delta\alpha\beta} + Q^{\delta\mu}\epsilon_{\mu}{}^{\alpha\beta\gamma}\right), \tag{80}$$

where we are using the usual square bracket notation for indices to denote antisymmetrization over the enclosed indices (e.g. $T^{\alpha[\delta}g^{\gamma]\beta} = \frac{1}{2}\left(T^{\alpha\delta}g^{\gamma\beta} - T^{\alpha\gamma}g^{\delta\beta}\right)$) and employing the rule that nested brackets are not operated on by enclosing brackets (e.g. $T^{[\alpha[\delta}g^{\gamma]\beta]} = \frac{1}{4}\left(T^{\alpha\delta}g^{\gamma\beta} - T^{\alpha\gamma}g^{\delta\beta} - T^{\beta\delta}g^{\gamma\alpha} + T^{\beta\gamma}g^{\delta\alpha}\right)$).

The other two tensors, $D'^{\alpha\beta\gamma\delta}$ and $X'^{\alpha\beta\gamma\delta}$, are however not completely reducible. So from them we can construct two new valence-4 tensors

$$D^{\alpha\beta\gamma\delta} :=D'^{\alpha\beta\gamma\delta} - 2\mathbf{i}D^{[\alpha[\delta}g^{\gamma]\beta]} - \frac{2}{3}L_{+}g^{\alpha[\delta}g^{\gamma]\beta} - \frac{1}{3}L_{-}\epsilon^{\alpha\beta\gamma\delta} \tag{81}$$
$$X^{\alpha\beta\gamma\delta} :=X'^{\alpha\beta\gamma\delta} - 2\mathbf{i}X^{[\alpha[\delta}g^{\gamma]\beta]} - \frac{2}{3}L_{-}g^{\alpha[\delta}g^{\gamma]\beta} + \frac{1}{3}L_{+}\epsilon^{\alpha\beta\gamma\delta} \tag{82}$$

Table I: Summary of a complete set of irreducible tensors. The heading “No. indep. comp.” stands for “number of independent components” . Note that the quantities in parentheses are alternatives elements of the set.

Concomitant	Total valence	No. indep. comp.
$\mathcal{L}_+$	0	1
$\mathcal{L}_-$	0	1
$T^{\alpha\beta}$	2	9
$\mathbf{i}Q^{\alpha\beta}$	2	9
$\mathbf{i}D^{\alpha\beta}$ (or $\mathbf{i}X^{\alpha\beta}$)	2	6
$D^{\alpha\beta\gamma\delta}$(or $X^{\alpha\beta\gamma\delta}$)	4	10

both of which fulfill (18), (19), and (20) and are therefore irreducible. Both $\mathcal{D}^{\alpha\beta\gamma\delta}$ and $\mathcal{X}^{\alpha\beta\gamma\delta}$ are real. They are, however, not independent of each other; in fact

$$X^{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu}D^{\mu\nu\gamma\delta}, \quad D^{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu}X^{\mu\nu\gamma\delta}, \tag{83}$$

that is, they are mutually dual over their two leftmost indices.

In a local, Lorentzian metric, coordinate system, the components of $D^{\alpha\beta\gamma\delta}$ and $X^{\alpha\beta\gamma\delta}$ are

$$D^{\alpha\beta\gamma\delta} \leftrightarrow D^{AB} = \frac{1}{2}\begin{pmatrix} \Re\left\{\bar{\mathbf{E}}\otimes\mathbf{E}-\bar{\mathbf{B}}\otimes\mathbf{B}\right\} - \frac{2}{3}\mathcal{L}_+\mathbf{1}_3 & \Re\left\{\bar{\mathbf{E}}\otimes\mathbf{B}+\bar{\mathbf{B}}\otimes\mathbf{E}\right\} - \frac{2}{3}\mathcal{L}_-\mathbf{1}_3 \\ \Re\left\{\bar{\mathbf{E}}\otimes\mathbf{B}+\bar{\mathbf{B}}\otimes\mathbf{E}\right\} - \frac{2}{3}\mathcal{L}_-\mathbf{1}_3 & -\Re\left\{\bar{\mathbf{E}}\otimes\mathbf{E}-\bar{\mathbf{B}}\otimes\mathbf{B}\right\} + \frac{2}{3}\mathcal{L}_+\mathbf{1}_3 \end{pmatrix} \tag{84}$$

$$X^{\alpha\beta\gamma\delta} \leftrightarrow X^{AB} = \frac{1}{2}\begin{pmatrix} -\Re\left\{\bar{\mathbf{E}}\otimes\mathbf{B}+\bar{\mathbf{B}}\otimes\mathbf{E}\right\} + \frac{2}{3}\mathcal{L}_-\mathbf{1}_3 & \Re\left\{\bar{\mathbf{E}}\otimes\mathbf{E}-\bar{\mathbf{B}}\otimes\mathbf{B}\right\} - \frac{2}{3}\mathcal{L}_+\mathbf{1}_3 \\ \Re\left\{\bar{\mathbf{E}}\otimes\mathbf{E}-\bar{\mathbf{B}}\otimes\mathbf{B}\right\} - \frac{2}{3}\mathcal{L}_+\mathbf{1}_3 & \Re\left\{\bar{\mathbf{E}}\otimes\mathbf{B}+\bar{\mathbf{B}}\otimes\mathbf{E}\right\} - \frac{2}{3}\mathcal{L}_-\mathbf{1}_3 \end{pmatrix} \tag{85}$$

when expressed in bivector indexing.

VIII. IRREDUCIBLE TENSORIAL SET

The irreducible tensors constructed in the previous section can be assembled into an real-irreducible tensorial set in the same spirit as Fano and Racah [8]. Such a set is given in Table I. They fulfill all the criteria itemized in Section II.

The tensors in Table I form a complete set by which we mean that any covariant hermitian form tensor concomitant of $F^{\alpha\beta}$ can be written as linear combinations (including arbitrary contractions) of $g^{\alpha\beta}$, $g^{\alpha\beta}g^{\gamma\delta}$, $\epsilon^{\alpha\beta\gamma\delta}$, and the set of tensors in Table I. In terms of the number of independent components, the tensors in the table constitute 36 independent components in total. This is exactly the number of unique bilinear combinations possible from the 6 complex components of the bivector if both real and imaginary parts are counted separately. In the real-valued bivector case, there are only 21 second-order combinations possible in total. This matches the total number of independent components in the set: $\mathcal{L}_+$, $\mathcal{L}_-$, $T^{\alpha\beta}$, $D^{\alpha\beta\gamma\delta}$ (or $X^{\alpha\beta\gamma\delta}$).

Note that a complete irreducible set of tensors bilinear in $F^{\alpha\beta}$ is not entirely unique. This is because, tensor $D^{\alpha\beta}$ is isomorphic with $X^{\alpha\beta}$ (its dual) and $D^{\alpha\beta\gamma\delta}$ is isomorphic with $X^{\alpha\beta\gamma\delta}$, that is, either $X^{\alpha\beta}$ or $X^{\alpha\beta\gamma\delta}$ could be used in place of $D^{\alpha\beta}$ or $D^{\alpha\beta\gamma\delta}$ respectively. Therefore the complete irreducible tensorial set in Table I is but one out of four of such sets possible.

We have throughout assumed that $F^{\alpha\beta}$ is an arbitrary complex bivector. If $F^{\alpha\beta}$ were real, then $\mathbf{i}Q^{\alpha\beta}$, $\mathbf{i}D^{\alpha\beta}$, $\mathbf{i}X^{\alpha\beta}$ are all zero; in other words, all the tensors in the Table with coefficient $\mathbf{i}$ vanish in the ordinary case of real-valued bivectors, leaving only $\mathcal{L}_+$, $\mathcal{L}_-$, $T^{\alpha\beta}$, $D^{\alpha\beta\gamma\delta}$, $X^{\alpha\beta\gamma\delta}$. Thus the existence of $\mathbf{i}Q^{\alpha\beta}$, $\mathbf{i}D^{\alpha\beta}$, $\mathbf{i}X^{\alpha\beta}$ is a direct consequence of the fact that an arbitrary complex $F^{\alpha\beta}$ can have an imaginary part.

IX. CONCLUSION

We have derived a complete set of irreducible tensorial concomitants that are bilinear in an arbitrary complex bivector. The set is summarized in Table I. The tensors in the set are all hermitian forms in the complex bivector $F^{\alpha\beta}$ and invariant, up to a sign, under the duality transformation of the bivector. That the valence-2 tensors are irreducible is equivalent them being trace-free, see Eq. (17), and that the valence-4 tensors are irreducible is equivalent to saying that they simultaneously fulfill the conditions (18), (19), and (20).

Of the tensors in the Table, $\mathcal{L}_+$,$\mathcal{L}_-$,$T^{\alpha\beta}$ are all hermitian form generalizations of previously known bilinear concomitants. All other tensors are, as far as the authors are aware, novel.

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